Modeling of a heat equation with a Dirac density

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Abstract

We consider a linear hybrid system consisting of two rods connected by a thin wall of width 2ϵ and density $1/2\epsilon$. By passing to a limit, we obtain a system describing heat flow of two rods connected by a "point mass" whose dynamics are governed by a differential equation. We prove that solutions of the *epsilon* problem converge weakly to solutions of the limiting system.

1 INTRODUCTION

Consider a linear hybrid system consisting of two wires or rods connected by a thin wall of width $2\epsilon > 0$ and density $1/2\epsilon$. Assume the two rods occupy the intervals $\omega_{\epsilon,1} = (-L_1, -\epsilon)$ and $\omega_{\epsilon,2} = (\epsilon, L_2)$, and the wall occupies the interval $\omega_\epsilon = (-\epsilon, \epsilon)$. Correspondingly, let $u_\epsilon = u_\epsilon(t,x)$, $v_\epsilon = v_\epsilon(t,x)$ and $z_\epsilon = z_\epsilon(t,x)$ denote the temperature distribution on their respective domains $\omega_{\epsilon,1}$, $\omega_{\epsilon,2}$, and ω_ϵ. We suppose the temperature of the rods and wall satisfy the heat equation on their respective domains with Dirichlet boundary conditions at endpoints $x = -L_1, L_2$. The linear equation modeling heat flow of such a system is as follows:

$$\begin{cases} c_1\rho_1 \dot{u_\epsilon} - k_1 {u_\epsilon}'' = 0, & t>0,\ x \in \omega_{\epsilon,1} \\ c_2\rho_2 \dot{v_\epsilon} - k_2 {v_\epsilon}'' = 0, & t>0,\ x \in \omega_{\epsilon,2} \\ \frac{c}{2\epsilon}\dot{z}_\epsilon - k {z_\epsilon}'' = 0, & t>0,\ x \in \omega_\epsilon \\ u_\epsilon(t,-\epsilon) = z_\epsilon(t,-\epsilon),\ z_\epsilon(t,\epsilon) = v_\epsilon(t,\epsilon), & t>0 \\ k_1 {u_\epsilon}'(t,-\epsilon) = k {z_\epsilon}'(t,-\epsilon),\ k{z_\epsilon}'(t,\epsilon) = k_2 {v_\epsilon}'(t,\epsilon), & t>0 \\ u_\epsilon(t,-L_1) = v_\epsilon(t,L_2) = 0, & t>0. \end{cases} \tag{1}$$

Throughout this article, $'$ will denote spatial derivatives and $\dot{}$ will denote temporal derivatives. The parameters $c > 0$ and $k > 0$ in the third equation represent the specific heat and conductivity of the wall connecting the two rods. The parameters c_i, ρ_i and k_i in (1) are positive and represent the specific heat, density and thermal conductivity of the rod on the subdomain $\omega_{\epsilon,i}$. It will later be convenient to use the diffusivity coefficient $\alpha_i^2 = k_i/c_i\rho_i$ for $i = 1, 2$. The fourth equation guarantees continuity of the temperature across the interface $x = \pm\epsilon$ and the fifth equation represents the heat flux continuity condition at the interfaces (see [13, Chapter 8]). We complete the system by adding the initial conditions

$$\{u_\epsilon^0(x), v_\epsilon^0(x), z_\epsilon^0(x)\} = \{u_\epsilon(0,x), v_\epsilon(0,x), z_\epsilon(0,x)\} \tag{2}$$

in an appropriately defined function space at time $t = 0$ so we may determine the solution of (1) uniquely.

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We show in this article that with appropriate assumptions on the initial conditions, the solution $\{u_\epsilon, v_\epsilon, z_\epsilon\}$ of (1) with (2) converges in a weak sense to the solution of the following limiting hybrid system:

$$\begin{cases} c_1\rho_1\dot{u} - k_1u'' = 0, & t>0,\ x\in\omega_1 := \omega_{0,1} \\ c_2\rho_2\dot{v} - k_2v'' = 0, & t>0,\ x\in\omega_2 := \omega_{0,2} \\ c\ \dot{z} = k_2v'(t,0) - k_1u'(t,0), & t>0 \\ u(t,0) = v(t,0) = z(t), & t>0 \\ u(t,-L_1) = v(t,L_2) = 0, & t>0, \end{cases} \tag{3}$$

with initial conditions of the form

$$\{u^0(x), v^0(x), z^0\} = \{u(0,x), v(0,x), z(0)\} \tag{4}$$

given in an appropriately defined function space at time $t=0$. The third equation in (3) states that the rate of change in temperature of the point mass is proportional to the net heat flux into the point mass. This can be viewed as a form of Fick's law of diffusion.

Similar hybrid systems involving strings and beams with point masses have been studied in the context of controllability and stabilization theory. See for example [8], [11], [2], [4], [10], [12], [5], [16], [7] and [6]. In particular, C. Castro showed in [3] that a system similar to (3) with strings can be obtained from a system similar to that in (1) and gave a detailed spectral analysis.

2 WELL-POSEDNESS

We begin by proving well-posedness of the limit problem (3).

2.1 The limit problem

Given u, v defined on ω_1, ω_2 and $z\in\mathbb{R}$, let $y=(u,v,z)^t$, where ^{t} denotes transposition and define

$$\mathcal{H} = L^2(\omega_1)\times L^2(\omega_2)\times\mathbb{R}$$

equipped with the norm

$$\|y\|^2_{\mathcal{H}} = c_1\rho_1\|u\|^2_{\omega_1} + c_2\rho_2\|v\|^2_{\omega_2} + c\ |z|^2$$

where $\|\cdot\|_{\omega_i}$ is the usual norm in $L^2(\omega_i)$ for $i=1,2$. Define

$$\begin{aligned} \vartheta_{\omega_1} &= \{u\in H^1(\omega_1)\mid u(-L_1)=0\} \\ \vartheta_{\omega_2} &= \{v\in H^1(\omega_2)\mid v(L_2)=0\} \\ \vartheta &= \{(u,v)\in\vartheta_{\omega_1}\times\vartheta_{\omega_2}\mid u(0)=v(0)\} \end{aligned}$$

equipped with the norms

$$\|u\|^2_{\vartheta_{\omega_i}} = k_i\|u'\|^2_{L^2(\omega_i)}, \quad \|(u,v)\|^2_{\vartheta} = \|u\|^2_{\vartheta_{\omega_1}} + \|v\|^2_{\vartheta_{\omega_2}}$$

for $i=1,2$. We can check that (see [8]) ϑ is algebraically and topologically equivalent to $H^1_0(\Omega)$, however one can think of ϑ as a subspace of $\vartheta_{\omega_1}\times\vartheta_{\omega_2}$. The space

$$\mathcal{W} = \{(u,v,z)\in\vartheta\times\mathbb{R}\mid u(0)=v(0)=z\}$$

is a closed subspace of $\vartheta \times \mathbb{R}$ with norm we may define as $\|y\|_{\mathcal{W}}^2 = \|(u,v)\|_{\vartheta}^2$. It is easy to see that the space $\mathcal{W}$ is densely and continuously embedded in the space $\mathcal{H}$. Define the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A} = \begin{pmatrix} \alpha_1^2 d^2 & 0 & 0 \\ 0 & \alpha_2^2 d^2 & 0 \\ -\frac{k_1}{c}\delta_0 d & \frac{k_2}{c}\delta_0 d & 0 \end{pmatrix} \tag{5}$$

where d denotes the (distributional) derivative operator and δ_0 denotes the Dirac delta function with mass at $x = 0$, and the domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{y \in \mathcal{W} \ : \ u \in H^2(\omega_1),\ v \in H^2(\omega_2)\}. \tag{6}$$

When $D(\mathcal{A})$ is endowed with the graph-norm topology

$$\|y\|_{D(\mathcal{A})}^2 = \|y\|_{\mathcal{H}}^2 + \|\mathcal{A}y\|_{\mathcal{H}}^2$$

it becomes a Hilbert space with continuous embedding in $\mathcal{H}$. We can therefore write the limit system (3) as

$$\dot{y}(t) = \mathcal{A}y(t), \quad y(0) = y^0, \quad t > 0 \tag{7}$$

where $y^0 = (u^0, v^0, z^0)$. Since $D(\mathcal{A})$ is dense in $\mathcal{W}$ and the latter is dense in $\mathcal{H}$, it follows that $\mathcal{A}$ is a densely defined operator.

Lemma 2.1.1. *The operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a bijection.*

Proof. Let $\vec{f} = (f, g, h) \in \mathcal{H}$ be arbitrary. Then the solution to $\mathcal{A}y = \vec{f}$ is given by

$$y = \begin{pmatrix} C_u(x + L_1) - F(x) \\ C_v(x - L_2) - G(x) \\ C_z \end{pmatrix} \tag{8}$$

where

$$\begin{aligned} F(x) &= \int_{-L_1}^{x} \int_{s}^{0} \alpha_1^{-2} f(r)\ drds, \\ G(x) &= \int_{x}^{L_2} \int_{0}^{s} \alpha_2^{-2} g(r)\ drds, \end{aligned} \qquad \begin{aligned} C_u &= \frac{-chL_2 + k_2(F(0) - G(0))}{k_2 L_1 + k_1 L_2} \\ C_v &= \frac{chL_1 + k_1(F(0) - G(0))}{k_2 L_1 + k_1 L_2} \\ C_z &= -\frac{chL_1L_2 + L_2k_1F(0) + L_1k_2G(0)}{k_2 L_1 + k_1 L_2}. \end{aligned} \tag{9}$$

Since $(u'', v'') = (\alpha_1^{-2} f, \alpha_2^{-2} g) \in L^2(\omega_1) \times L^2(\omega_2)$ it follows that $(u, v) \in H^2(\omega_1) \times H^2(\omega_2)$. Furthermore, one can check from (9) that $u(0) = v(0) = z$ and $u(-L_1) = v(L_2) = 0$ so that $y \in D(\mathcal{A})$. Thus $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is surjective.

Finally, note that the null space of $\mathcal{A}$ is trivial since when $\vec{f} = (0, 0, 0)$ we see that y is the trivial solution. Then $\mathcal{A}$ is injective and hence bijective. $\square$

Lemma 2.1.2. *The operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is symmetric and dissipative.*

Proof. Consider $\varphi = (\mu, \nu, \zeta) \in D(\mathcal{A})$. Then

$$\begin{aligned}
\langle \mathcal{A}y, \varphi\rangle_{\mathcal{H}} &= k_1 u'\mu|_{-L_1}^{0} - k_1\langle u', \mu'\rangle_{\omega_1} + k_2 v'\nu|_0^{L_2} - k_2\langle v', \nu'\rangle_{\omega_2} + (k_2 v'(0) - k_1 u'(0))\zeta \\
&= -k_1 u\mu'|_{-L_1}^{0} + k_1\langle u, \mu''\rangle_{\omega_1} - k_2 v\nu'|_0^{L_2} + k_2\langle v, \nu''\rangle_{\omega_2} \\
&= c_1\rho_1\langle u, \alpha_1^2\mu''\rangle_{\omega_1} + c_2\rho_2\langle v, \alpha_2^2\nu''\rangle_{\omega_2} + cz\left(\frac{k_2}{c}\nu'(0) - \frac{k_2}{c}\mu'(0)\right) \\
&= \langle y, \mathcal{A}\varphi\rangle_{\mathcal{H}}
\end{aligned}$$

for all $y = (u, v, z) \in D(\mathcal{A})$. Hence $D(\mathcal{A}) \subset D(\mathcal{A}^*)$ and so $\mathcal{A}$ is a symmetric operator. In particular, when we choose $y = \varphi$ we see from the above computation that

$$\langle \mathcal{A}y, y\rangle_{\mathcal{H}} = -k_1\|u'\|_{\omega_1}^2 - k_2\|v'\|_{\omega_2}^2 = -\|y\|_{\mathcal{W}}^2.$$

Thus we have that $\langle \mathcal{A}y, y\rangle_{\mathcal{H}} \leq 0$ for any $y \in D(\mathcal{A})$ as needed to show $\mathcal{A}$ is dissipative. $\square$

Lemma 2.1.3. *The operator $\mathcal{A}$ is closed, self-adjoint and its inverse is a compact operator in $\mathcal{H}$.*

Proof. As mentioned before, $\mathcal{A}$ is densely defined in $\mathcal{H}$ and from Lemmas 2.1.2 and 2.1.1 we have that it is symmetric and $R(\mathcal{A}) = \mathcal{H}$. It follows from Theorem 13.11 in [15], that $\mathcal{A}$ is self-adjoint and its inverse $\mathcal{A}^{-1}$ is bounded in $\mathcal{H}$. Furthermore, since the inverse is bounded, we have $0 \in \rho(\mathcal{A})$ and Theorem 13.9 in [15] implies that $\mathcal{A}$ is closed.

Next we claim that $K := \mathcal{A}^{-1}$ is compact. From formulas (8)-(9) we can decompose $K = K_1 + K_2$ where

$$K_1\vec{f} = \begin{pmatrix} -F(x) \\ -G(x) \\ 0 \end{pmatrix}, \quad K_2\vec{f} = \begin{pmatrix} C_u(x + L_1) \\ C_v(x - L_2) \\ C_z \end{pmatrix}.$$

Since the mappings $f \mapsto F(x)$ and $g \mapsto G(x)$ are Volterra-type operators, K_1 is compact. Since $0 \in \rho(\mathcal{A})$ and K_1 is compact, K_2 must be bounded. Since also K_2 has finite rank, it follows that K_2 is compact, and hence also K is compact. $\square$

Proposition 2.1.1. *The operator $\mathcal{A}$ is the infinitesimal generator of a strongly continuous C_0-semigroup of contractions which extends for $Re(t) > 0$ to an analytic semigroup.*

Proof. By Lemma 2.1.3, we have $\mathcal{A}$ is a closed, densely defined and self-adjoint operator and by Lemma 2.1.2 we see that both $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative. Therefore, by the Lümer-Phillips theorem (see Luo et al [9]), we have that $\mathcal{A}$ generates a C_0-semigroup of contractions.

Furthermore, the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is contained in $(-\infty, 0)$ and by the computations shown in [1, page 55], we obtain $\|R(z, \mathcal{A})\| \leq \sec(\theta/2)/|z|$ for all $z = \rho e^{i\theta} \in \mathbb{C} \setminus (-\infty, 0]$ and $\theta \in (\pi/2, \pi)$. Rewriting the angle θ as $\pi/2 + \delta$ where $0 < \delta < \pi/2$ and letting $M = \sec(\theta/2)$ we see that

$$\|R(z, \mathcal{A})\| \leq \frac{M}{|z|} \text{ for all } z \in \mathbf{S}_\theta$$

where

$$\mathbf{S}_\theta = \{z \in \mathbb{C} \;:\; |\arg z| < \theta\}.$$

Note that $\mathbf{S}_\theta \cup \{0\}$ is contained in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$. By Theorem 5.2. in [14] we have that T can be extended to an analytic semigroup in a sector $\mathbf{S}_\delta$. If $\mathrm{Re}(z) > 0$ then for $\delta < \pi/2$ large enough, that $z \in \mathbf{S}_\delta$. Hence A generates an analytic semigroup in the right half plane $\mathrm{Re}(z) > 0$. $\square$

As a consequence of Proposition 2.1.1, given initial data $y^0 \in \mathcal{H}$, there exists a unique solution

$$y \in C([0,\infty);\mathcal{H}) \tag{10}$$

to the Cauchy problem (7). If in addition, $y^0 \in D(\mathcal{A})$, then $y \in C([0,\infty);D(\mathcal{A}))$.

2.2 The approximate problem

Now consider functions u_ϵ, v_ϵ and z_ϵ defined on $\omega_{\epsilon,1}$, $\omega_{\epsilon,2}$ and ω_ϵ respectively and define $y_\epsilon = (u_\epsilon, v_\epsilon, z_\epsilon)^t$. Let

$$\mathcal{H}_\epsilon = L^2(\omega_{\epsilon,1}) \times L^2(\omega_{\epsilon,2}) \times L^2(\omega_\epsilon).$$

equipped with the norm

$$\|y_\epsilon\|^2_{\mathcal{H}_\epsilon} = c_1\rho_1\|u_\epsilon\|^2_{\omega_{\epsilon,1}} + c_2\rho_2\|v_\epsilon\|^2_{\omega_{\epsilon,2}} + \frac{c}{2\epsilon}\|z_\epsilon\|^2_{\omega_\epsilon}$$

where $\|\cdot\|_{\omega_{\epsilon,i}}$ is the usual norm in $L^2(\omega_{\epsilon,i})$ for $i = 1,2$ and $\|\cdot\|_{\omega_\epsilon}$ is the usual norm in $L^2(\omega_\epsilon)$. Define

$$\vartheta_{\omega_{\epsilon,1}} = \{u_\epsilon \in H^1(\omega_{\epsilon,1}) \mid u_\epsilon(-L_1) = 0\}$$
$$\vartheta_{\omega_{\epsilon,2}} = \{v_\epsilon \in H^1(\omega_{\epsilon,2}) \mid v_\epsilon(L_2) = 0\}$$

equipped with the norms

$$\|u_\epsilon\|^2_{\vartheta_{\omega_{\epsilon,i}}} = k_i\|u_\epsilon{}'\|^2_{L^2(\omega_{\epsilon,i})},$$

for $i = 1,2$. Next, consider the following subspace of $\vartheta_{\omega_{\epsilon,1}} \times \vartheta_{\omega_{\epsilon,2}} \times H^1(\omega_\epsilon)$:

$$\mathcal{W}_\epsilon = \{y_\epsilon \in \vartheta_{\omega_{\epsilon,1}} \times \vartheta_{\omega_{\epsilon,2}} \times H^1(\omega_\epsilon) \mid u_\epsilon(-\epsilon) = z_\epsilon(-\epsilon),\ z_\epsilon(\epsilon) = v_\epsilon(\epsilon)\}$$

with the norm

$$\|y_\epsilon\|^2_{\mathcal{W}_\epsilon} = k_1\|u_\epsilon{}'\|^2_{\omega_{\epsilon,1}} + k_2\|v_\epsilon{}'\|^2_{\omega_{\epsilon,2}} + k\|z_\epsilon{}'\|^2_{\omega_\epsilon}.$$

Remark 2.2.1. It is easy to show that the spaces $\mathcal{W}_\epsilon$ are uniformly equivalent to $H^1_0(\Omega)$ in the sense that there exists some constant $C > 0$ such that

$$\frac{1}{C}\|\varphi\|_{\mathcal{W}_\epsilon} \leq \|\varphi\|_{H^1_0(\Omega)} \leq C\|\varphi\|_{\mathcal{W}_\epsilon}$$

where C is independent of ϵ for all $0 < \epsilon < \epsilon_0$ with finite ϵ_0. Furthermore, it is easy to see that the space W_ϵ is densely and continuously embedded in the space $\mathcal{H}_\epsilon$.

We will also make use of the space $\mathcal{H}^2_\epsilon = H^2(\omega_{\epsilon,1}) \times H^2(\omega_{\epsilon,2}) \times H^2(\omega_\epsilon)$. Define the unbounded operators $\mathcal{A}_\epsilon : D(\mathcal{A}_\epsilon) \subset \mathcal{H}_\epsilon \to \mathcal{H}_\epsilon$ by

$$\mathcal{A}_\epsilon = \begin{pmatrix} \alpha_1^2 d^2 & 0 & 0 \\ 0 & \alpha_2^2 d^2 & 0 \\ 0 & 0 & \frac{2\epsilon k}{c} d^2 \end{pmatrix}, \tag{11}$$

with domain $D(\mathcal{A}_\epsilon)$ given by

$$D(\mathcal{A}_\epsilon) = \left\{ y_\epsilon \in \mathcal{W}_\epsilon \;:\; y_\epsilon \in \mathcal{H}^2_\epsilon,\ k_1 {u_\epsilon}'(-\epsilon) = k{z_\epsilon}'(-\epsilon),\ k{z_\epsilon}'(\epsilon) = k_2 {v_\epsilon}'(\epsilon) \right\}.$$

When $D(\mathcal{A}_\epsilon)$ is equipped with the graph-norm topology

$$\|y_\epsilon\|^2_{D(\mathcal{A}_\epsilon)} = \|y_\epsilon\|^2_{\mathcal{H}_\epsilon} + \|\mathcal{A}_\epsilon y_\epsilon\|^2_{\mathcal{H}_\epsilon},$$

it becomes a Hilbert space with continuous embedding in $\mathcal{H}_\epsilon$. We can now rewrite system (1) as a Cauchy problem:

$$\dot{y_\epsilon}(t) = \mathcal{A}_\epsilon y_\epsilon(t), \quad y_\epsilon(0) = y^0_\epsilon \in \mathcal{H}_\epsilon, \quad t > 0. \tag{12}$$

It is easy to see that $\mathcal{A}_\epsilon$ is densely defined on $\mathcal{H}_\epsilon$. As in Section 2.1 we have the following results.

Lemma 2.2.1. *The operator $\mathcal{A}_\epsilon : D(\mathcal{A}_\epsilon) \to \mathcal{H}_\epsilon$ is a bijective, dissipative, closed, self-adjoint operator with a compact inverse in $\mathcal{H}_\epsilon$.*

Proposition 2.2.1. *The operator $\mathcal{A}_\epsilon$ is the infinitesimal generator of a strongly continuous C_0-semigroup of contractions which extends for $Re(t) > 0$ to an analytic semigroup.*

The fact that $\mathcal{A}_\epsilon$ is the infinitesimal generator of an analytic C_0-semigroup implies that for all $y^0_\epsilon \in \mathcal{H}_\epsilon$ there exists a unique solution

$$y_\epsilon \in C([0,\infty); \mathcal{H}_\epsilon) \tag{13}$$

to (12). Moreover, if $y^0_\epsilon \in D(\mathcal{A}_\epsilon)$, then also $y_\epsilon \in C([0,\infty); D(\mathcal{A}_\epsilon))$.

3 WEAK CONVERGENCE

The energy functional of the hybrid system (3) is given by $E(t) = \|y\|^2_{\mathcal{H}}/2$. By taking test functions $\varphi \in C^1_0\big([0,\infty)\times\Omega\big)$, a weak form of the hybrid system (3) is given by

$$\begin{aligned}
&\int_{\omega_1} c_1\rho_1 u^0 \varphi(0,x)\ dx + \int_{\omega_2} c_2\rho_2 v^0 \varphi(0,x)\ dx + cz^0\varphi(0,0) \\
&\qquad = -\int_0^\infty \left\{ \int_{\omega_1} c_1\rho_1 u\dot{\varphi}\ dx + \int_{\omega_2} c_2\rho_2 v\dot{\varphi}\ dx + cz\dot{\varphi}(t,0) \right\} dt \\
&\qquad + \int_0^\infty \left\{ \int_{\omega_1} k_1 u'\varphi'\ dx + \int_{\omega_2} k_2 v'\varphi'\ dx \right\} dt.
\end{aligned} \tag{14}$$

On the other hand, the energy functional for the ϵ-dependent problem (1) is $E_\epsilon(t) = \|y_\epsilon\|^2_{\mathcal{H}_\epsilon}/2$ and by taking test functions $\varphi \in C^1_0\big([0,\infty)\times\Omega\big)$, a weak form is

$$\begin{aligned}
&\int_{\omega_{\epsilon,1}} c_1\rho_1 u^0_\epsilon \varphi(0,x)\ dx + \int_{\omega_{\epsilon,2}} c_2\rho_2 v^0_\epsilon \varphi(0,x)\ dx + \int_{\omega_\epsilon} \frac{c}{2\epsilon} z^0_\epsilon \varphi(0,x)\ dx \\
&\qquad = -\int_0^\infty \left\{ \int_{\omega_{\epsilon,1}} c_1\rho_1 u_\epsilon\dot{\varphi}\ dx + \int_{\omega_{\epsilon,2}} c_2\rho_2 v_\epsilon\dot{\varphi}\ dx + \int_{\omega_\epsilon} \frac{c}{2\epsilon} z_\epsilon\dot{\varphi}\ dx \right\} dt \\
&\qquad + \int_0^\infty \left\{ \int_{\omega_{\epsilon,1}} k_1 {u_\epsilon}'\varphi'\ dx + \int_{\omega_{\epsilon,2}} k_2 {v_\epsilon}'\varphi'\ dx + \int_{\omega_\epsilon} k{z_\epsilon}'\varphi'\ dx \right\} dt.
\end{aligned} \tag{15}$$

We give sufficient conditions such that we may pass to the limit in (15) to consequently obtain (14). Assume that $y_\epsilon^0 \in D(\mathcal{A}_\epsilon)$ and furthermore, there exists $M_1 > 0$ such that

$$\|y_\epsilon^0\|_{\mathcal{H}_\epsilon} \le M_1, \tag{16}$$

for all $\epsilon > 0$. Then we obtain the following result.

Lemma 3.0.2. *The energy of system* (1) *is (uniformly) bounded by the initial energy in the sense that there exists constant $C > 0$ such that $E_\epsilon(t) \le C$ for all $\epsilon > 0$ whenever the initial data y_ϵ^0 satisfies* (16).

Proof. Note that if $y_\epsilon^0 \in D(\mathcal{A}_\epsilon)$, then $y_\epsilon \in C\big([0,\infty); D(\mathcal{A}_\epsilon)\big)$ and the energy satisfies

$$\begin{aligned}\dot{E}_\epsilon(t) &= k_1 {u_\epsilon}' u_\epsilon\big|_{-L_1}^{-\epsilon} + k_2 {v_\epsilon}' v_\epsilon\big|_{\epsilon}^{L_2} + k {z_\epsilon}' z_\epsilon\big|_{-\epsilon}^{\epsilon} \\ &\qquad - \int\limits_{\omega_{\epsilon,1}} k_1 |{u_\epsilon}'|^2 \, dx - \int\limits_{\omega_{\epsilon,2}} k_2 |{v_\epsilon}'|^2 \, dx - \int\limits_{\omega_\epsilon} k |{z_\epsilon}'|^2 \, dx \\ &= -\|y_\epsilon\|_{\mathcal{W}_\epsilon}^2,\end{aligned} \tag{17}$$

which implies $\dot{E}_\epsilon(t) \le 0$, and thus $E_\epsilon(t) \le E_\epsilon(0)$. Hence by density, there exists $C = M_1^2/2$ for which $E_\epsilon(t) \le C$ for all $t > 0$ and initial data satisfying (16). Consequently, we find that $y_\epsilon \in L^\infty\big([0,\infty); \mathcal{H}_\epsilon\big)$ for all $\epsilon > 0$. ☐

Now assume there exists $M_2 > 0$ such that

$$\|y_\epsilon^0\|_{\mathcal{W}_\epsilon} \le M_2, \tag{18}$$

for all $\epsilon > 0$. Then we obtain the following result.

Lemma 3.0.3. *Assuming condition* (18) *holds, the sequence solutions $\{y_\epsilon\}_{\epsilon>0}$ to problem* (1) *is uniformly bounded in $L^\infty\big([0,\infty); H_0^1(\Omega)\big)$.*

Proof. Since $y_\epsilon^0 \in D(\mathcal{A}_\epsilon)$ we have that $y_\epsilon \in C\big([0,\infty); D(\mathcal{A}_\epsilon)\big)$ and

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|y_\epsilon\|_{\mathcal{W}_\epsilon}^2 &= k_1 \dot{u}_\epsilon {u_\epsilon}'\big|_{-L_1}^{-\epsilon} + k_2 \dot{v}_\epsilon {v_\epsilon}'\big|_{\epsilon}^{L_2} + k \dot{z}_\epsilon {z_\epsilon}'\big|_{-\epsilon}^{\epsilon} \\ &\qquad - k_1 \langle \dot{u}_\epsilon, {u_\epsilon}'' \rangle_{\epsilon,1} \, dx - k_2 \langle \dot{v}_\epsilon {v_\epsilon}'' \rangle_{\epsilon,2} - k \langle \dot{z}_\epsilon {z_\epsilon}'' \rangle_\epsilon \\ &= -\|{y_\epsilon}'\|_{\mathcal{W}_\epsilon}^2.\end{aligned}$$

This shows that the sequence $\{\|y_\epsilon(t)\|_{\mathcal{W}_\epsilon}^2\}_{t>0}$ is monotone decreasing in t and thus by density, $\|y_\epsilon(t)\|_{\mathcal{W}_\epsilon} \le \|y_\epsilon^0\|_{\mathcal{W}_\epsilon} \le M_2$ for all $t > 0$ as needed to show $y_\epsilon \in L^\infty(0,\infty; \mathcal{W}_\epsilon)$. From Remark 2.2.1 we see that the spaces $\mathcal{W}_\epsilon$ and $H_0^1(\Omega)$ are equivalent and thus there exists $K > 0$ independent of ϵ such that

$$\|y_\epsilon\|_{L^\infty([0,\infty); H_0^1(\Omega))} \le K$$

for all $\epsilon > 0$, as needed to show solutions to the ϵ-dependent problem are uniformly bounded in $\mathcal{Y}$. ☐

Next, it is natural to assume the initial data convergences in a weak sense in $\mathcal{H}_\epsilon$. It is easy to see that u_ϵ^0 being a sequence in $L^2(\omega_{\epsilon,1})$ implies that $\chi_{\omega_{\epsilon,1}} u_\epsilon^0$ is a sequence in $L^2(\omega_1)$. Likewise, $\chi_{\omega_{\epsilon,2}} v_\epsilon^0 \in L^2(\omega_1)$. Then we will assume that

$$\begin{cases} \chi_{\omega_{\epsilon,1}} u_\epsilon^0 \rightharpoonup u_0 \text{ weakly in } L^2(\omega_1) \text{ as } \epsilon \to 0 \\ \chi_{\omega_{\epsilon,2}} v_\epsilon^0 \rightharpoonup v_0 \text{ weakly in } L^2(\omega_2) \text{ as } \epsilon \to 0 \\ \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon^0 \, dx \to z_0 \text{ in } \mathbb{R} \text{ as } \epsilon \to 0. \end{cases} \tag{19}$$

The following theorem is our main result.

Theorem 3.0.1. *Let $\{y_\epsilon\}_{\epsilon>0}$ be the sequence of solutions to the $\epsilon-$dependent problem* (1) *with initial data y^0_ϵ. Assuming* (16)*,* (18) *and* (19)*, the family $\{y_\epsilon\}_{\epsilon>0}$ of solutions to* (1) *problem satisfies*

$$y_\epsilon \overset{*}{\rightharpoonup} y \text{ in } L^\infty\big([0,\infty; H^1_0(\Omega)\big)$$

as $\epsilon \to 0$ where y is the weak solution to the limit problem (3) *with initial $y^0 \in \mathcal{H}$.*

Proof. From Lemmas 3.0.2 and 3.0.3, the initial energy provides a uniform bound for the solutions $y_\epsilon \in L^\infty\big([0,\infty; H^1_0(\Omega)\big)$. We can then extract a subsequence of solutions (which is still denoted by the index ϵ) such that

$$\chi_{\omega_{\epsilon,1}} u_\epsilon \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,\infty; L^2(\omega_1)) \cap L^\infty(0,\infty; \vartheta_{\omega_1}) \tag{20}$$

$$\chi_{\omega_{\epsilon,2}} v_\epsilon \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,\infty; L^2(\omega_2)) \cap L^\infty(0,\infty; \vartheta_{\omega_2}). \tag{21}$$

Next, observe that $g_\epsilon(t) := \frac{1}{2\epsilon}\langle 1, z_\epsilon(t)\rangle_\epsilon$ defines a function on $[0,\infty)$. Applying Holder's inequality we see that $|g_\epsilon(t)| \le \|z_\epsilon\|_\epsilon/\sqrt{2\epsilon}$. By condition (16) we have that $\{g_\epsilon\}_{\epsilon>0}$ is a uniformly bounded sequence in $L^\infty(0,\infty)$. Invoking the Banach-Alaoglu Theorem we can extract a subsequence of z_ϵ (still denoted with the index ϵ) and find $z \in L^\infty(0,\infty)$ such that

$$g_\epsilon \overset{*}{\rightharpoonup} z \text{ in } L^\infty(0,\infty) \tag{22}$$

as $\epsilon \to 0$. We now pass to the limit in each of the nine terms in the characterization (15) of weak solutions of the ϵ-problem with $\varphi \in C^1_0([0,\infty)\times\Omega)$. Since $\varphi(0,\cdot) \in C^1_0(\Omega)$ it follows from assumption (19) on the initial data y^0_ϵ that

$$\int_{\omega_{\epsilon,1}} u^0_\epsilon \varphi(0,x)\ dx = \int_{\omega_1} \chi_{\omega_{\epsilon,1}} u^0_\epsilon \varphi(0,x)\ dx \to \int_{\omega_1} u^0 \varphi(0,x)\ dx,$$

$$\int_{\omega_{\epsilon,2}} v^0_\epsilon \varphi(0,x)\ dx = \int_{\omega_2} \chi_{\omega_{\epsilon,2}} v^0_\epsilon \varphi(0,x)\ dx \to \int_{\omega_2} v^0 \varphi(0,x)\ dx.$$

Next we claim that $\int_{\omega_\epsilon} \frac{1}{2\epsilon} z^0_\epsilon \varphi(0,x)\ dx \to z^0\varphi(0,0)$. By adding and subtracting the term $z^0\varphi(0,x)$ and applying the triangle inequality we find that

$$\left|\int_{\omega_\epsilon} \frac{1}{2\epsilon} z^0_\epsilon \varphi(0,x)\ dx - z^0\varphi(0,0)\right| \le \max_{x\in\omega_\epsilon}|\varphi(0,x)| \left|\frac{1}{2\epsilon}\int_{\omega_\epsilon} z^0_\epsilon - z^0\ dx\right| + |z^0|\frac{1}{2\epsilon}\int_{\omega_\epsilon} |\varphi(0,x) - \varphi(0,0)|\ dx.$$

The first term in the right hand side above tends to zero from (19) and the boundedness of $\varphi(0,\cdot)$ on ω_ϵ. The second term tends to zero as well by the continuity of $\varphi(0,x)$.

From (20) and (21) we have that in particular for $\dot\varphi \in C\big([0,\infty)\times\Omega\big)$ that

$$\int_0^\infty \int_{\omega_{\epsilon,1}} u_\epsilon \dot\varphi\ dx\ dt \to \int_0^\infty \int_{\omega_1} u\dot\varphi(t,x)\ dx\ dt,$$

$$\int_0^\infty \int_{\omega_{\epsilon,2}} v_\epsilon \dot\varphi\ dx\ dt \to \int_0^\infty \int_{\omega_2} v\dot\varphi(t,x)\ dx\ dt.$$

Next we want to show that

$$\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon(t,x)\dot\varphi(t,x)\ dx\ dt \to \int_0^\infty z(t)\dot\varphi(t,0)\ dt. \tag{23}$$

Observe that

$$
\begin{aligned}
&\left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon(t,x)\dot\varphi(t,x)\ dx\ dt - \int_0^\infty z(t)\dot\varphi(t,0)\ dx\ dt\right| \\
&\qquad = \left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon\dot\varphi(t,x) - z\dot\varphi(t,0)\ dx\ dt\right| \\
&\qquad \le \left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon(\dot\varphi(t,x)-\dot\varphi(t,0))\ dx\ dt\right| + \left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} (z_\epsilon - z)\dot\varphi(t,0)\ dx\ dt\right|.
\end{aligned}
$$

The last term tends to zero by (22). Regarding the first term, observe that applying Hölder's inequality we have

$$
\begin{aligned}
\left|\int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} z_\epsilon(\dot\varphi(t,x)-\dot\varphi(t,0))\ dx\ dt\right| &\le \int_0^\infty \frac{1}{2\epsilon}\int_{\omega_\epsilon} |z_\epsilon|\,|\dot\varphi(t,x)-\dot\varphi(t,0)|\ dx\ dt \\
&\le \int_0^\infty \frac{1}{\sqrt{2\epsilon}}\|z_\epsilon\|_\epsilon \frac{1}{\sqrt{2\epsilon}}\sqrt{\int_{\omega_\epsilon} |\dot\varphi(t,x)-\dot\varphi(t,0)|\ dx}\ dt \\
&\le \int_0^\infty \frac{1}{\sqrt{2\epsilon}}\|z_\epsilon\|_\epsilon \sqrt{\frac{1}{2\epsilon}\int_{\omega_\epsilon} |\dot\varphi(t,x)-\dot\varphi(t,0)|\ dx}\ dt.
\end{aligned}
$$

Note that $\|z_\epsilon\|_\epsilon/\sqrt{2\epsilon}$ is bounded by condition (16) and by the continuity of $\dot\varphi$, we have that the above tends to zero as $\epsilon \to 0$. Since $\varphi' \in C([0,\infty)\times\Omega)$ we have from (20) and (21) that

$$
\begin{aligned}
\int_0^\infty \int_{\omega_{\epsilon,1}} {u_\epsilon}'\varphi'\ dx\ dt &= \int_0^\infty \int_{\omega_1} \chi_{\omega_{\epsilon,1}} {u_\epsilon}'\varphi'\ dx\ dt \to \int_0^\infty \int_{\omega_1} u'\varphi'\ dx\ dt \\
\int_0^\infty \int_{\omega_{\epsilon,2}} {v_\epsilon}'\varphi'\ dx\ dt &= \int_0^\infty \int_{\omega_2} \chi_{\omega_{\epsilon,2}} {v_\epsilon}'\varphi'\ dx\ dt \to \int_0^\infty \int_{\omega_2} v'\varphi'\ dx\ dt.
\end{aligned}
$$

Finally, the term $\int_0^\infty \int_{\omega_\epsilon} k{z_\epsilon}'\varphi'\ dxdt$ tends to zero as $\epsilon \to 0$ since $z_\epsilon(t,\cdot) \in H^1(\omega_\epsilon)$ and $\varphi' \in C([0,\infty)\times\Omega)$.

From the above discussion we now have that there exists a convergent subsequence of y_ϵ in the weak star sense, whose limit satisfies the equation (14). Since the limiting system has a unique weak solution, it follows that the convergence holds for the whole sequence $\{y_\epsilon\}_{\epsilon>0}$. Therefore, we have shown that the limiting system (3) can be approximated with the sequence of ϵ dependent problems (1) as needed. □

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